Investigations on the Plane Quartic.

By Teresa Cohen.

§ 1. *Introduction.*

The plane quartic $(\alpha x)^4$ is taken in the form

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3$$
$$+12mx_0x_1^2x_2+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4.$$

It will be convenient first to mention briefly certain well-known forms connected with it that will be made use of in this article.

Of these several arise from the polar forms, $(\alpha x)^3(\alpha y)$, $(\alpha x)^2(\alpha y)^2$, $(\alpha x)(\alpha y)^3$. Since of each of these there is an ∞^2, the placing of one, two, or three conditions on the curves they represent results, respectively, in a locus for the pole, in a set of points, and in an invariant condition to be satisfied by the quartic.

Upon the polar line the only condition that can be imposed is its identical vanishing. This, as is well known, means that the quartic has a double point and requires the vanishing of the discriminant, an A^{27}.*

The polar conic may be made to break up into two lines. The locus of poles of such degenerate conics is the Hessian H, an A^3x^6. To make these two lines coincide requires two additional conditions and gives for the quartic an invariant, shown by Dr. Thomsen † to be an A^{48}.

From the polar cubic more can be obtained. The locus of poles of polar cubics having a double point is the Steinerian Σ, an $A^{12}x^{12}$. The cubic has also the invariants S and T, of degrees 4 and 6, respectively, which give rise to the covariants of the quartic $S\equiv A^4x^4$ and $T\equiv A^6x^6$. From the relation connecting the discriminant of the cubic with these two invariants we have

$$\Sigma\equiv 64S^3+T^2,$$

a very useful form for calculating coefficients of the Steinerian whenever that may be necessary. To require that the polar cubic have a cusp is two conditions, giving rise to a set of twenty-four points with cuspidal polar cubics. The number of these points is determined by the fact that they are common

* The notation $A^ix^j\xi^k$ is used to represent a comitant form of degree i in the coefficients of the quartic, j in x, and k in ξ.

† American Journal of Mathematics, Vol. XXXVIII (1916), p. 249.

points of S and T; therefore they are cusps of the Steinerian. There are also twenty-one points whose polar cubics have two double points and therefore break up into a line and a conic; these are the double points of the Steinerian. The line parts of these cubics will hereafter be referred to simply as "the twenty-one lines." Each of these lines meets the quartic so that the four tangents at the intersections are all on a point, which point is the corresponding point; *i. e.*, the pole of the cubic of which the line forms a part. If one of these lines be taken as $x_0=0$ and the corresponding point as $(1, 0, 0)$ then in $(\alpha x)^4$

$$b_0=m=n=c_0=0.$$

The quartic also has certain contravariants obtained by imposing a condition on the four points in which a line cuts it. The locus of lines cutting the quartic in a self-apolar set is $s\equiv(s\xi)^4\equiv A^2\xi^4$. The locus of lines cutting in harmonic pairs is $t\equiv(t\xi)^6\equiv A^3\xi^6$.

§ 2. *The Undulation.*

If a line cuts the quartic in four consecutive points, the quartic is said to have an undulation. The invariant vanishing in this case is given by Salmon as an A^{60}. The undulation tangent is evidently a line of both s and t, and it is that special case of the twenty-one lines occurring when one of the lines is on its corresponding point. This corresponding point is the undulation itself, which is therefore a double point of Σ. Suppose the undulation to be at $(0, 1, 0)$ with the tangent x_0. Then

$$b=b_2=f=c_1=0.$$

The undulation is also a point of the Hessian, for its polar conic is

$$hx_0^2+2b_0x_0x_1+2mx_0x_2=x_0(hx_0+2b_0x_1+2mx_2),$$

a pair of lines. Let the double point of this conic be taken as $(0, 0, 1)$, which therefore becomes the corresponding point on the Steinerian. Then

$$m=0.$$

The polar cubic of $(0, 1, 0)$ is

$$a_1x_0^3+3hx_0^2x_1+3lx_0^2x_2+3b_0x_0x_1^2+3nx_0x_2^2.$$

This is made up of x_0 and a conic; its two double points are $(0, \sqrt{n}, \sqrt{-b_0})$ and $(0, \sqrt{n}, -\sqrt{-b_0})$; they are the two points of the Hessian corresponding to the double point of the Steinerian and are harmonic to the undulation point and to the Steinerian point corresponding to the undulation considered as a Hessian point. The terms in H not containing x_0 are

$$x_1^4x_2^2\cdot -cb_0^2+x_1^2x_2^4\cdot -2cb_0n+x_2^6\cdot -cn^2=-cx_2^2(b_0x_1^2+nx_2^2)^2.$$

Therefore x_0 is a triple tangent to the Hessian.

§ 3. *The Discriminant of the Hessian.*

The Hessian and Steinerian, as is well known, are not independent curves, but are brought into one-to-one correspondence by the relation

$$(\alpha x)(\alpha y)^2 \alpha_i = 0, \qquad i = 0, 1, 2,$$

where x is a point of the Steinerian and y a point of the Hessian. This says that the polar cubic of x has a double point y and the polar conic of y a double point x. The joins of all such pairs x and y give rise to the Cayleyan, an $A^{12}\xi^{18}$. Since x and y can not come together unless

$$(\alpha x)^3 \alpha_i = 0, \qquad i = 0, 1, 2,$$

which is the condition that the quartic have a double point, there is a very convenient reference scheme for handling these three curves in any other case. Let $(0, 1, 0)$ be a point of the Hessian, $(0, 0, 1)$ the corresponding point on the Steinerian, so that x_0 is a line of the Cayleyan. Then

$$\alpha_1^2 \alpha_2 \alpha_i = 0, \qquad i = 0, 1, 2,$$

or

$$m = b_2 = f = 0.$$

Now let us see under what conditions the Hessian acquires a double point. It is known to have one when the quartic does. To discover other cases let us use the above reference scheme. Then

$$H \equiv x_1^5 x_0 \cdot 2n(bh - b_0^2) + x_1^5 x_2 \cdot 2c_1(bh - b_0^2) + \text{lower terms in } x_1.$$

Therefore $(0, 1, 0)$ is a double point if

$$bh - b_0^2 = 0.$$

But this says that the polar conic of $(0, 1, 0)$, which is $hx_0^2 + 2b_0 x_0 x_1 + bx_1^2$, shall be a line counted twice, and this means the vanishing of an A^{48}. Since this line is already on $(0, 0, 1)$, let it be taken as x_1, so that

$$b_0 = h = 0.$$

Then the polar cubic of any point $(k, 0, 1)$ on x_1, namely

$$(ak + a_2)x_0^3 + 3(a_1 k + l)x_0^2 x_1 + 3(a_2 k + g)x_0^2 x_2 + 6(lk + n)x_0 x_1 x_2 \\ + 3(gk + c_0)x_0 x_2^2 + 3(nk + c_1)x_1 x_2^2 + (c_0 k + c)x_2^3,$$

has a double point at $(0, 1, 0)$; therefore the Steinerian must contain the line x_1. Two of these polar cubics have cusps, the poles of which may be made $(1, 0, 0)$ and $(0, 0, 1)$ by requiring that

$$l = n = 0.$$

The cuspidal tangent of each cubic is on the pole of the other, and the pair of them make up the tangents to the Hessian at its double point. Then

$$S\equiv -a_1^2c_1^2x_0^2x_2^2+x_0^3x_1(-ba_1a_2c_0+ba_1g^2)+\ldots+x_1x_2^3(-ba_2c_0c_1+bc_1g^2) + \text{higher powers in } x_1,$$

and

$$T\equiv 8a_1^3c_1^3x_0^3x_2^3+x_0^5x_1\cdot 4ba_1^3c_0^2+\ldots+x_1x_2^5\cdot 4ba_2^2c_1^3+ \text{higher powers in } x_1.$$

Therefore S touches x_1 and T has x_1 as a flex line at (1, 0, 0) and (0, 0, 1). Also

$$\Sigma\equiv x_1[x_0^8x_2^3\cdot 64ba_1^6c_0^2c_1^3+\ldots+x_0^3x_2^8\cdot 64ba_1^3a_2^2c_1^6 +x_1\{x_0^{10}\cdot 16b^2a_1^6c_0^4+\ldots+x_2^{10}\cdot 16b^2a_2^4c_1^6\}+ \text{higher powers in } x_1],$$

showing that x_1 divides out only once, but that (1, 0, 0) and (0, 0, 1) are more than ordinary singularities.

The Hessian also has a double point when

$$c_1=n=0.$$

The symmetry shows that (0, 0, 1) is also a double point, as can be verified from the coefficients. Then (0, 1, 0) as a Hessian point has (0, 0, 1) as the corresponding Steinerian point, and vice versa. The polar conics of (0, 1, 0) and (0, 0, 1) are, respectively,

$$hx_0^2+2b_0x_0x_1+bx_1^2 \text{ and } gx_0^2+2c_0x_0x_2+cx_2^2.$$

If the harmonic conjugate of x_0 as to the first pair of lines is taken as x_1 and that as to the second as x_2, then

$$b_0=c_0=0.$$

This shows that x_0 is one of the twenty-one lines. To find the degree of the invariant vanishing in this case is one of the objects of this investigation. With the above reference scheme

$$S\equiv x_1x_2^3\cdot -bcgl+x_2^2\{x_0^2(cgh^2-chl^2)+x_0x_1\cdot -bca_1g+x_1^2(-bcgh-bcl^2)\} + \text{lower powers in } x_2,$$

while

$$T\equiv x_1x_2^5\cdot 2bc^2l^3+x_2^4\{x_0^2\cdot -3c^2h^2l^2+2x_0x_1(-3bc^2a_2hl+6bc^2a_1l^2) +x_1^2(b^2c^2a_2^2+4b^2cg^3+12bc^2hl^2)\}+ \text{lower powers in } x_2.$$

Therefore these two curves touch at (0, 0, 1) along the line x_1, and similarly at (0, 1, 0) along the line x_2. Then

$$\Sigma\equiv x_1^2x_2^{10}\cdot 4b^2c^4l^6+x_2^9\{x_0^2x_1\cdot -12bc^4h^2l^5+x_0x_1^2\cdot 8bc^2l^3(-3bc^2a_2hl+6bc^2a_1l^2) +x_1^3\cdot 4bc^2l^3(b^2c^2a_2^2-12b^2cg^3+12bc^2hl^2)\}+ \text{lower powers in } x_2.$$

Therefore Σ has a singularity at (0, 0, 1), and, symmetrically, at (0, 1, 0), which is something more than merely a cusp. It may be a tac-node.

These three cases make up the totality of ways in which the Hessian may acquire a double point. Therefore its discriminant, an A^{225}, must be made up simply of the three invariants attached to them.

§ 4. *The Discriminant of* $(s\xi)^4$.

The contravariant $(s\xi)^4$ is on the line x_0 if

$$b_2=f=c=0.$$

The point of contact is given by

$$3c_1m\xi_1+(-bc_0+b_0c_1)\xi_2=0.$$

Since (0, 0, 1) is taken as any one of the points in which x_0 cuts the quartic, this point of contact will be on $(\alpha x)^4$ if $m=0$ (if $c_1=0$, the quartic would have an undulation). But this says that (0, 0, 1) is a point of the Steinerian. Therefore the forty-eight intersections of $(\alpha x)^4$ and $(s\xi)^4$, which considered as a point curve is of order twelve, are the same as those of $(\alpha x)^4$ and Σ. This can be substantiated by finding the point equation of $(s\xi)^4$. The line equation of $(\alpha x)^4$ is s^3-27t^2. But s formed for $(s\xi)^4$ is $-12S+A^3(\alpha x)^4$, where A^3 is the invariant A given by Salmon, and t formed for $(s\xi)^4$ is

$$T+(\alpha x)^4\cdot(t\alpha')^3(t\alpha'')^3(\alpha' x)(\alpha'' x).$$

Therefore the line equation of $(s\xi)^4$ is

$$[-12S+A^3(\alpha x)^4]^3-27[T+(\alpha x)^4\cdot(t\alpha')^3(t\alpha'')^3(\alpha' x)(\alpha'' x)]^2$$
$$=-27\Sigma+(\alpha x)^4\cdot A^{11}x^8.$$

x_0 will become a double line of s only if

$$b_0=c_0=m=0,$$

all other conditions than this placing more than one restriction on the quartic. The points in which x_0 meets the quartic are (0, 0, 1), (0, k, 1), (0, ωk, 1), (0, $\omega^2 k$, 1), where $k^3=-\dfrac{4c_1}{b}$ and $\omega^2+\omega+1=0$. The tangents to the quartic at these four points are, respectively,

$$\begin{aligned} c_1x_1 &=0,\\ 3nkx_0-3c_1x_1+3c_1kx_2 &=0,\\ 3n\omega kx_0-3c_1x_1+3c_1\omega kx_2 &=0,\\ 3n\omega^2kx_0-3c_1x_1+3c_1\omega^2kx_2 &=0, \end{aligned}$$

and they evidently have the common point $(c_1, 0, -n)$. Let $n=0$, so that this common point becomes (1, 0, 0). Since among the coefficients equated to zero are b_2, m, f, then (0, 1, 0) is a point of the Hessian with (0, 0, 1) as its corresponding Steinerian point. Since the tangent to the Steinerian at any

point is the polar line as to the quartic of the corresponding Hessian point, the tangent to the Steinerian at (0, 0, 1) is x_1, which is the tangent to $(\alpha x)^4$ at the same point. Then since no distinction was made among the four points in which x_0 cuts the quartic by assigning the particular coördinates (0, 0, 1) to one of them, when one of the twenty-one lines is a line of $(s\xi)^4$, the quartic and Steinerian touch four times on this line, the four tangents going through the corresponding point. Also since the tangent to the Hessian at (0, 1, 0) is x_2, the tangents at the four Hessian points corresponding to these Steinerian points are also on that corresponding point.

x_0 will also be a line of $(s\xi)^4$ if

$$b=b_2=f=0;$$

i. e., if x_0 is a stationary line of the quartic. But this leads to nothing new. Therefore $(s\xi)^4$ has a double line only when one of the twenty-one lines is on $(s\xi)^4$, and its discriminant, an A^{54}, expresses this condition.

§ 5. *The Discriminant of* $(t\xi)^6$.

The contravariant $(t\xi)^6$ is on the line x_0 if

$$b=b_2=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot 2b_0c_1^2+\text{ lower powers in }\xi_0.$$

x_0 becomes a double line either if $b_0=0$, which says that the quartic has a double point at (0, 1, 0) with x_0 as a tangent there, or if $c_1=0$, which says that the quartic has an undulation.

$(t\xi)^6$ is also on x_0 when

$$b_2=c_1=f=0.$$

Then

$$(t\xi)^6=2\xi_0^5\xi_1\cdot -bcn+2\xi_0^5\xi_2\cdot - bcm+\text{ lower powers in }\xi_0.$$

For x_0 to be a double line requires either that $b=0$ or $c=0$, which repeats the undulation condition, or that $m=n=0$, which is the condition on the quartic under which the Hessian had two double points and for which the invariant of unknown degree vanishes. Then one of the twenty-one lines is a line of $(t\xi)^6$.

$(t\xi)^6$ also has x_0 as a line when $b=f=c=0$, but this leads to nothing new. Therefore the discriminant of $(t\xi)^6$, an A^{225}, must be made up of the discriminant of the quartic, an A^{27}, the undulation condition, an A^{60}, and the unknown invariant.

§ 6. *The Twenty-One Lines.*

The twenty-one lines * are given by an $A^i x^{21}$, where i is as yet unknown. That it can be determined is due to the fact that these lines are part of the common lines of two curves, all of whose common lines are known. These curves are the Cayleyan, an $A^{12}\xi^{18}$, and an $A^{15}\xi^{24}$, which is the locus of lines cutting the quartic so that three of the tangents at the intersections are on a point.† The common lines of these curves are are: ‡ (1) the twenty-one lines counted sixteen times, since they are quadruple lines of both curves; (2) the twenty-four stationary lines, given by an $A^{24}x^{24}$, counted twice because they are double lines of $A^{15}\xi^{24}$; (3) the forty-eight lines of $(s\xi)^4$ at its intersections with the quartic. Since the polar points of these lines are on $(\alpha x)^4$, they are lines of the curve $(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$, an $A^9\xi^{12}$, and are obtained by the elimination of ξ from $A^9\xi^{12}$, $(s\xi)^4 \equiv A^2\xi^4$, and (ξx) as an $A^{60}x^{48}$. The eliminant of $A^{12}\xi^{18}$, $A^{15}\xi^{24}$, and (ξx) is an $A^{558}x^{432}$, which must be made up of the common lines together with such invariants as express the fact that they become indeterminate. The only such invariant is the discriminant of the quartic, for if the quartic has a double point it divides out at least once from each of these curves.

$$A^{558}x^{432} = (A^i x^{21})^{16} \cdot (A^{24}x^{24})^2 \cdot A^{60}x^{48} \cdot (A^{27})^k.$$
$$\therefore\ 558 = 16i + 108 + 27k, \quad 16i + 27k = 450.$$

This equation shows that i must be a multiple of 9, and it can be satisfied only by $i = 18$. Therefore the twenty-one lines are given by an $A^{18}x^{21}$.

§ 7. *The Invariant of Unknown Degree.*

The degree of the form giving the twenty-one lines being known, we can force one of them to lie on either $(s\xi)^4$ or $(t\xi)^6$. The condition so obtained will contain the undulation condition, however, for the tangent at an undulation is a special case of one of the twenty-one lines, and is a line of both s and t. Suppose the $A^{18}x^{21}$ factored into its component lines, the coördinates of each of

---

* A curve on these lines is

$$A^9\xi^{12} \equiv 2(s\xi)^3(sa)(s'\xi)^3(s'a)(s''\xi)^3(s''a)(s'''\xi)^3(s'''a) - 3(s\xi)^4(s'\xi)^3(s'a)(s''\xi)^3(s''a)(s'''\xi)^2(s'''a)^2 + \tfrac{1}{3}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''a)^4.$$

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 227.

‡ Let $b_2 = m = f = 0$, so that x_0 is a line of the Cayleyan. Let $n = 0$, so that x_2 is the tangent to the Hessian at (0, 1, 0). The tangent to the Steinerian can not be chosen as the other reference line because the stationary lines are known to be among the common lines of the two curves and the Steinerian is tangent to them. Instead, let (1, 0, 0) be determined by the intersection of x_2 with the polar line of (0, 0, 1). Then $c_0 = 0$. The coefficient of ξ_0^{24} in $A^{15}\xi^{24}$ becomes $-\frac{10}{3}b^2cb_0^4c_1^8$. If $b = 0$, x_0 is a stationary line of the quartic. If $c = 0$, the quartic and Σ meet at (0, 0, 1) and x_0 is a line of $(s\xi)^4$. If $b_0 = 0$, x_0 is one of the twenty-one lines. If $c_1 = 0$, an invariant condition is imposed on the quartic.

these substituted in $(s\xi)^4$, the twenty-one resulting terms multiplied together, and the coefficients of the $A^{18}x^{21}$ substituted for symmetric functions of these coördinates. The result is an

$$A^{18 \cdot 4 + 2 \cdot 21} = A^{114} = A^{60} \cdot A^{54}.$$

This leads to nothing new. But the same process applied to $(t\xi)^6$ gives the degree of the required invariant. For we obtain $A^{18 \cdot 6 + 21 \cdot 3} = A^{171}$ from which the undulation condition must divide out at least once. Therefore the invariant in question may be either of degree 111 or 51. The first of these will not fit into the discriminant of the Hessian, but the second would give

$$A^{225} = A^{27} \cdot (A^{48})^2 \cdot (A^{51})^2,$$

while for the discriminant of $(t\xi)^6$ we should have

$$A^{225} = (A^{27})^2 \cdot (A^{60})^2 \cdot A^{51}.$$

Therefore we have an invariant of degree 51, the vanishing of which expresses the fact that the Hessian has two double points, that $(t\xi)^6$ has a double line, and that one of the twenty-one lines is a line of $(t\xi)^6$. It also has another meaning, as will appear later.

§ 8. *Certain other Invariants.*

The degree of the form giving the twenty-one lines being known, that for the twenty-one corresponding points can be calculated from the fact that each point is the polar point of the corresponding line as to both $(s\xi)^4$ and $(t\xi)^6$. Suppose, as before, the $A^{18}x^{21}$ factored into its component lines, the coördinates substituted for η in $(s\xi)(s\eta)^3$, and the symmetric functions of these coördinates in the product of the resulting terms replaced by the coefficients of $A^{18}x^{21}$. The result gives the twenty-one points together with such invariants as express that the polar point becomes indeterminate (*i. e.*, that one of the lines be a double line of s). Then

$$A^{18 \cdot 3 + 2 \cdot 21}\xi^{21} = A^{54} \cdot A^{42}\xi^{21}.$$

Similarly from $(t\xi)^6$ we obtain

$$A^{18 \cdot 5 + 3 \cdot 21}\xi^{21} = A^{60} \cdot A^{51} \cdot A^{42}\xi^{21}.$$

Therefore the twenty-one points are given by an $A^{42}\xi^{21}$.* Suppose one of them to be on $(\alpha x)^4$, and therefore on its own polar cubic. Then, as before, considering $A^{42}\xi^{21}$ broken up into its component points, the coördinates substituted

* This degree fits in with the form $A^{36}\xi^{18}$ for the line equation of Σ. For the cusps are obtained from the eliminant of S, T, and (ξx) as an $A^{48}\xi^{24}$. Then from the Plücker formula

$$A^{12 \cdot 22}\xi^{12 \cdot 11} = A^{264}A^{132} = (A^{42}\xi^{21})^2 \cdot (A^{48}\xi^{24})^3 \cdot A^{36}\xi^{18}.$$

in $(\alpha x)^4$, and their symmetric functions replaced in the product of the resulting terms, we obtain an $A^{42 \cdot 4+21}=A^{189}$. Out of this the condition for an undulation, when the point is on the line part of its polar cubic, must divide out at least once. If it is no more than once, we have left an A^{129} to express that the point is on the conic part of its polar cubic. To substantiate this degree we must find some case into which the undulation condition does not enter. We had before that the point is (1, 0, 0) and its corresponding line x_0 if $b_0=m=n=c_0=0$. To put the point on the quartic requires that $a=0$. The undulation case is excluded because our choice of coördinates for the point and line implies that they are not incident. We can still choose (0, 1, 0) and (0, 0, 1) on x_0. Let us take them as the double points of the polar cubic of (1, 0, 0), so that $g=h=0$. Then in $(s\xi)^4$ the coefficients of ξ_1^4, ξ_2^4, $\xi_1^3\xi_2$, $\xi_1\xi_2^3$ vanish, showing that x_1 and x_2 are lines of s with contact (1, 0, 0), so that (1, 0, 0) is a double point of $(s\xi)^4$.* It is known that the double lines of $(\alpha x)^4$ are given by an $A^{16}x^{28}$; therefore the double points of $(s\xi)^4$ are given by an $A^{32}\xi^{28}$. One of them can lie on $(\alpha x)^4$ only in our present case or when $(\alpha x)^4$ has a double point, for them $(s\xi)^4$ has that same double point with the same tangents. Then by the method of considering the $A^{32}\xi^{28}$ factored into its component parts we have as the condition that one of these points be on $(\alpha x)^4$ an

$$A^{32 \cdot 4+28}=A^{156}=A^{129} \cdot A^{27}.$$

This establishes the degree of the A^{129}.

The totality of polar cubics of the twenty-one points given by the $A^{42}\xi^{21}$ is given by an $A^{63}x^{63}$. Suppose this factored into its components cubics $(\beta_i x)^3$, and then each of these factored into its component line and conic $(\gamma_i x) \cdot (\delta_i x)^2$. Then the condition that the line touch the conic is of second degree in both γ_i and δ_i, therefore of second degree in β_i. Therefore the invariant expressing the condition that there be a polar cubic made up of a conic and a line touching it is an $A^{63 \cdot 2}=A^{126}$. This result can be substantiated in other ways. If, as usual, we take the line as x_0 and the corresponding point as (1, 0, 0) so that

$$b_0=m=n=c_0=0,$$

then the conic will touch x_0 at (0, 1, 0) if

$$h=l=0.$$

Then S goes through (1, 0, 0) while T has a double point there, so that Σ has two cusps and a double point coming together and acquires a triple point, since x_0^9 is the highest power of x_0 appearing in its equation. Since two of the

*An examination of the coefficients of $(s\xi)^4$ shows that an undulation is not a double point of it.

intersections of S and T have come together, the A^{126} must be a factor of the tact-invariant of these two curves. We have seen (§ 3) that in case of the vanishing of either the A^{48} or the A^{51} that S and T touch at two distinct points, and these three cases seem to be the only ones in which two intersections of these curves come together. Their tact-invariant is found from the formula given by Salmon to be of degree $4\cdot 6(6+8-3)+6\cdot 4(4+12-3)=576$. Then

$$A^{576}=(A^{48})^2\cdot(A^{51})^2\cdot(A^{126})^3.$$

Also, inspection of their coefficients shows that both H and S touch x_0 at $(0, 1, 0)$. As will be seen later, two intersections of S and H can come together only when two of S and T do. When the A^{51} vanishes H has two double points on S; when the A^{48} vanishes, H has a double point which is also a double point of S. The tact-invariant of S and H is of degree $4\cdot 6(6+8-3)+3\cdot 4(4+12-3)=420$.

$$A^{420}=(A^{48})^4\cdot(A^{51})^2\cdot A^{126}.$$

The twenty-one polar cubics being given by an $A^{63}x^{63}$, while the twenty-one line parts are given by an $A^{18}x^{21}$, the twenty-one conic parts are given by an $A^{45}x^{42}$. Since the discriminant of a conic is of third degree in its coefficients, the condition that one of these conics break up is an A^{135}. But then the polar cubic consists of three lines, any two of which may be considered as the conic part, so that this degree is probably divided by three. Therefore the condition that there be a polar cubic made up of three lines is an A^{45}. Then three of the double points of the Steinerian have come together, giving it a triple point.*

§ 9. *The Eliminant of S, T, H.*

Under the reference scheme where

$$b_2=f=m=0,$$

the coefficient of the highest term in x_2 in both S and T is a power of (c_1l+n^2). Then both coefficients vanish if $c_1=n=0$, but this, as we have seen, means the

---

* Professor Morley has pointed out to me that the degrees of these last two invariants can be substantiated by the work of Caporali (Memorie di Geometria, p. 171) on a web of plane curves. If i curves of the web are of a particular kind, then the invariant expressing that there be a curve of that kind in a net is of degree i in the coefficients of the three curves upon which the net is built up. From his results the invariants expressing that in a net of cubics there be a cubic with three double points or a cubic with two coincident double points are of degree 15 and 42, respectively, in the coördinates of the three cubics. But in the polar net each cubic is linear in the coefficients of the quartic. Therefore the invariants are are of degree 45 and 126 in the coefficients of the quartic.

vanishing of an A^{51}. Therefore for the general quartic we obtain (0, 0, 1) as one of the common points of S and T if we make

$$l=n=0.$$

But then the coefficient of x_1^4 vanishes in S. Therefore to the twenty-four cusps of the Steinerian correspond those twenty-four points of the Hessian which make up its intersection with S.

It is known that the polar cubic of a point on S can be reduced to the sum of the cubes of three linear factors; its Hessian is the product of these factors and is one of those four triangles whose sides pass through all the flexes of the cubic. If the cubic has a double point, there is only one such proper triangle, made up of the tangents at the double point and a line through the three remaining flexes (flex line); in case of a cusp even this degenerates to the cusp tangent counted twice and the line joining the cusp to the sole remaining flex. Here the polar cnbic of (0, 0, 1) is

$$a_2x_0^3+3gx_0^2x_2+3c_0x_0x_2^2+3c_1x_1x_2^2+cx_2^3,$$

and its Hessian is

$$-c_1^2x_2^2(a_2x_0+gx_2).$$

But x_2 is the tangent to H at (0, 1, 0) and $a_2x_0+gx_2$ is the tangent to S at the same point. Therefore the polar cubic of a cusp of the Steinerian has as its cusp tangent and its flex line the tangents to H and S, respectively, at the corresponding point. The cubic can be thrown into the form

$$\frac{1}{a_2^2}\left[(a_2x_0+gx_2)^3+x_2^2\{3a_2(a_2c_0-g^2)x_0+3a_2^2c_1x_1+(ca_2^2-g^3)x_2\}\right]$$

which also shows up the tangent at the flex as the linear form in brackets.

The polar cubic of (0, 1, 0) is

$$a_1x_0^3+3hx_0^2x_1+3b_0x_0x_1^2+bx_1^3+c_1x_2^3$$

and its Hessian is

$$c_1x_2[x_0^2(a_1b_0-h^2)+x_0x_1(ba_1-b_0h)+x_1^2(bh-b_0^2)]$$

Then the tangent to H at (0, 1, 0) passes through three flexes of the cubic, the three flex tangents pass through the corresponding Steinerian point and the binary Hessian of these three tangents picks up the six remaining flexes.

The coefficient of x_2^6 in H is $-c_1^2g$. Since we cannot have $c_1=0$ without bringing on the vanishing of the A^{51} and worse, let us make $g=0$. Then we have made S, T, H have a common point at (0, 0, 1). The polar cubic of (0, 0, 1) becomes

$$a_2x_0^3+x_2^2(3c_0x_0+3c_1x_1+cx_2).$$

If we let the intersection of stationary tangent and cusp tangent be (1, 0, 0), then

$$c_0 = 0.$$

But since we now have $g = n = c_0$, the Steinerian point corresponding to (0, 0, 1) as a Hessian point is (1, 0, 0), which also becomes a cusp on the Steinerian. So to sum up, we have: (1) the cusp e_0 on the Steinerian corresponds to e_2 on the Hessian; (2) the cusp e_2 on the Steinerian corresponds to e_1 on the Hessian; (3) the tangent to the Hessian at e_1 goes through e_0 and at e_2 goes through e_1; (4) the polar cubic of e_2 has the tangent to the Hessian at e_1 as cusp tangent, the line $x_0 (= e_1 e_2)$ as flex line, and a stationary tangent going through e_0; (5) the polar cubic of e_0 has an analogous set of lines; (6) S is tangent to x_0 at e_1 and to x_1 at e_2.

The real underlying significance of this state of affairs is not known; neither is the degree of the invariant representing it, but it can be guessed at. For the eliminant of S, T, H is an A^{360}. The only other condition under which these three curves have a common point is when the A^{51} vanishes, but then they have four common points, for H has two double points on the other curves. Then presumably the A^{51} divides four times out of the eliminant, leaving an A^{156}. The essential difference between the two cases seems to be this: To every ST point (intersection of S and T) there is attached an SH point with which it cannot coincide without more than one condition on the quartic. In order for S, H, T to have a common point an ST point must coincide with an SH point. If then the corresponding SH and ST points also coincide, the A^{51} vanishes. When there is no further coincidence, we have the other case.

§ 10. *The Eliminant of the Polar Cubic, Conic, and Line.*

Suppose we ask that the polar cubic, conic, and line of (0, 1, 0), which is not on the quartic, have a common point (0, 0, 1). Then

$$b_2 = f = c_1 = 0.$$

The symmetry of this condition shows that the relation between (0, 1, 0) and (0, 0, 1) is a mutual one. Also x_0 is a line of $(t\xi)^6$. Therefore we might define $(t\xi)^6$ as the locus of the joins of pairs of points so related that the polar curves of each pass through the other, the pair of points being the common harmonic pair of the two harmonic pairs of points in which their join cuts the quartic.

In general the eliminant of the polar curves of x, which are an Ax^3y, an Ax^2y^2, and an Axy^3, is an $A^{11}x^{26}$. Out of this $(\alpha x)^4$ divides twice, leaving an

A^9x^{18}. Before asking the meaning of this latter curve, let us examine the eliminant of a cubic Ay^3, its polar conic Axy^2, and its polar line Ax^2y. It is an $A^{11}x^{15}$, from which the cubic divides out twice. The remainder, an A^9x^9, gives the nine stationary lines. Therefore the A^9x^{18} is for the quartic the locus of points lying on the stationary lines of their polar cubics, also the locus of flexes of polar cubics the tangent at which lies on the pole.

Salmon gives as the equation of the nine stationary lines of the cubic.

$$5SU^2H-H^3-U\theta,$$

where S is the invariant of degree 4, U is the cubic itself, H is its Hessian and θ is a certain A^8x^6. For a polar cubic of the quartic the first three pass over readily enough into the covariant S, the quartic itself, and its Hessian, but θ is not so easily transferred. So we will substitute for it another A^8x^6, expressible in terms of S, U, H, and θ, which gives the locus of a point whose polar conic as to the cubic is on its polar line as to the Hessian. Then it is easily shown that

$$A^9x^{18}=8SU^2H-H^3+9U\theta',$$

where S is the covariant S of the quartic, U the quartic itself, H its Hessian, and θ' the locus of those points whose polar conics as to the quartic are on their polar lines as to the Hessian.

The form of the A^9x^{18} shows that all of its intersections with $(\alpha x)^4$ are used up at the flexes of the quartic. The twenty-four points are also part of its intersection with H, but there are eighty-four more, lying on θ'. Now we saw that $(0, 1, 0)$ is a point of the A^9x^{18} if

$$b_2=f=c_1=0.$$

To make it a point of H also requires that

$$m=0.*$$

But there are only forty-two such points, for they are the points where H, $(t\xi)^6$ and the Cayleyan all touch.† Then the intersections of H and θ' must come together by twos, and, indeed, it is readily shown that the two curves touch at their common points.

It is also easily shown that the forty-two inflexions of the Steinerian are at the points corresponding to these Hessian points.

* Or symmetrically, $(0, 0, 1)$ is a point of H if $n=0$.

† Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.

§ 11. *The Polar Conic of Two Points.*

The eliminant of the polar curves of the quartic connects up with the polar conic of two points in the following way. The polar conic of x and y as to the quartic is $(\alpha x)(\alpha y)(\alpha z)^2$. Any point on the line xy may be represented parametrically as $x+\lambda y$. This line meets the conic in two points whose parameters λ are given by

$$(\alpha x)(\alpha y)(\alpha\overline{x+\lambda y})^2=(\alpha x)^3(\alpha y)+2\lambda(\alpha x)^2(\alpha y)^2+\lambda(\alpha x)(\alpha y)^3=0.$$

If we ask that these points become indeterminate, we ask that the coefficients of λ all vanish. Then x and y are a pair of points such that the polar curves of each are on the other, so that we can define the A^9x^{18} of the last section as the locus of pairs of points whose polar conic is not only degenerate but contains their join, and $(t\xi)^6$ as the locus of this join. If x and y be taken as (0, 1, 0) and (0, 0, 1), then, as we have seen,

$$b_2=f=c_1=0.$$

The polar conic of the two points is

$$lx_0^2+2mx_0x_1+2nx_0x_2=x_0(lx_0+2mx_1+2nx_2).$$

To make the double point at one of the points requires that

$$m=0 \quad (\text{or } n=0).$$

But this makes (0,1,0) one of the forty-two points on H where it touches $(t\xi)^6$.

In general for a given y we have a whole locus of points x, the polo-Hessian of y to be exact, such that the polar conic of x and y is a pair of lines. We have seen that for one of these lines to be xy, x and y must lie on an A^9x^{18}. If we ask instead that the double point be at x, then x is a point of the Hessian and y a point of the Steinerian. For the polar conic of (0,1,0) and (0,0,1) is

$$lx_0^2+2mx_0x_1+2nx_0x_2+b_2x_1^2+2fx_1x_2+c_1x_2^2,$$

and this becomes two lines on (0, 1, 0) if

$$m=b_2=f=0,$$

being, in fact, the tangents at the double point of the polar cubic of the Steinerian point. To ask both that xy be one of the lines and that the double point be at x picks out, as we have seen, the forty-two inflexions of the Steinerian, and the forty-two points where the Hessian touches the Cayleyan and $(t\xi)^6$.

Making a fresh start, let us ask that the polar conic of (0,1,0) and (0,0,1) be the square of the line. If we take (1, 0, 0) as a point of this line, then

$$l=m=n=0,\quad b_2c_1-f^2=0.$$

But then we find that the coefficients of x_1^4 and x_2^4 in S vanish. Therefore the pair of points whose polar conic is a repeated line lie on S. But we already have a correspondence of the points of S, for the polo-Hessian of a point of S is made up of three lines, the intersections of which are on S. With the above reference scheme

$$S_{400}=(a_1b_0-h^2)(-a_2c_0+g^2).$$

(1, 0, 0) was any point of the repeated line. Let us take it as one of the intersections with S by making

$$a_1b_0-h^2=0.$$

Then the polo-Hessian of (0, 1, 0) becomes

$$x_1[x_0x_1(ba_1f-a_1b_2^2-b_0fh)+x_0x_2(ba_1c_1-a_1b_2f-b_0c_1h)$$
$$+x_1^2(bfh-b_0^2f-b_2^2h)+x_1x_2(bc_1h-b_0^2c_1-b_2fh)].$$

The double points are (1, 0, 0), (0, 0, 1), and

$$y\equiv(bc_1h-b_0^2-b_2fh,\ -ba_1c_1+a_1b_2f+b_0c_1h,\ ba_1f-a_1b_2^2-b_0fh).$$

Then the polar conic of (0, 1, 0) and (1, 0, 0) becomes $a_1x_0^2+2hx_0x_1+b_0x_1^2$, that of (0, 1, 0) and (0, 0, 1) is $b_2x_1^2+2fx_1x_2+c_1x_2^2$, and that of (0, 1, 0) and y is x_1^2. Therefore the polar conic of a point on S and any vertex of its polo-Hessian triangle is the square of the opposite side.*

If, now, we also ask that the repeated line be on (0, 1, 0) we have

$$b_2=f=0.$$

Then (0, 1, 0) is one of the twenty-four intersections of S and H. Therefore the polar conic of an SH point and its corresponding ST point is the square of a line on the SH point (the tangent to H there).

If, finally, we ask that the polar conic of (0, 1, 0) and (0, 0, 1) be x_0^2, we have

$$m=n=b_2=f=c_1=0.$$

But then the A^{51} vanishes. Therefore we have a new definition for the A^{51}; it is the invariant expressing the condition that there be a pair of points such that their polar conic as to the quartic is the square of the line joining them.

§ 12. *Salmon's Connex.*

Salmon has shown that with any plane curve $(\alpha x)^n$ there is associated an $A^3x^{2(n-2)}y^{n-2}$, which when x is a point on the curve picks out the remaining

* We might expect this sort of (3, 1) correspondence from the following considerations: The line equation of $(\alpha x)(\alpha y)(\alpha z)^2$ is an $A^2x^2y^2\xi^2$. If the conic is the square of a line, the line equation vanishes identically. Equating each coefficient to zero, we have six equations from which we can eliminate the six quantities y_0^2, y_0y_1, y_0y_2, y_1^2, y_1y_2, y_2^2, giving an $A^{12}x^{12}$. This must be S^3.

intersections of the tangent at x with the curve. This connex is expressible in terms of polars of the Hessian of the curve itself and the Hessians of the polar curves of x as to $(\alpha y)^n$. In case of the quartic the connex is an $A^3x^4y^2 \equiv 15(hx)^4(hy)^2 - 9(h_1x)(h_1y)^2$, where $(hy)^6$ is the Hessian of the quartic and $(h_1y)^3$ is the Hessian of the polar cubic of x. Explicitly

$$\begin{aligned}
A^3x^4y^2 = \overset{3}{\Sigma} x_0^4 [& y_0^2 \cdot 6(agh - al^2 - a_1^2 g - a_2^2 h + 2a_1a_2l) \\
& + \overset{2}{\Sigma} y_0y_1 \cdot 4(ab_0g + ahn - 2alm - a_2^2b_0 - a_1^2 n + 2a_1a_2m - a_1gh + a_1l^2) \\
& + \overset{2}{\Sigma} y_1^2 (abg - ba_2^2 + ab_0n - 2ab_2l + afh - am^2 + 2a_1a_2b_2 - a_1^2 f - a_1b_0g \\
& \quad - a_1hn + 2a_1lm) \\
& + y_1y_2(ab_0c_0 + 2ab_2g + 2ac_1h - 4afl - amn - 2a_1^2c_1 - 2a_2^2b_2 \\
& \quad - 4a_1a_2f - a_1c_0h - a_2b_0g - a_1gm - a_2hn + 2a_1ln + 2a_2lm)] \\
+ \overset{6}{\Sigma} x_0^3x_1 [& y_0^2 \cdot 8(ab_0g + ahn - 2alm - a_2^2b_0 - a_1^2n + 2a_1a_2m - a_1gh + a_1l^2) \\
& + y_0y_1 \cdot 2(abg - ba_2^2 + 7ab_0n - 2ab_2l + afh - 7am^2 + 2a_1a_2b_2 - a_1^2 f \\
& \quad + 5a_1b_0g - 12a_2b_0l - 7a_1hn + 12a_2hm + 2a_1lm - 6gh^2 + 6hl^2) \\
& + y_0y_2 \cdot (7ab_0c_0 + 2ab_2g + 2ac_1h - 4afl - 7amn - 2a_1^2c_1 - 2a_2^2b_2 \\
& \quad + 4a_1a_2f - 7a_1c_0h - 7a_2b_0g + 17a_1gm + 17a_2hn - 10a_1ln \\
& \quad - 10a_2lm - 12ghl + 12l^3) \\
& + y_1^2 \cdot 3(abn + ab_0f + ba_1g - 2ab_2m - 2ba_2l + 2a_2b_2h - a_1fh - b_0gh \\
& \quad - hn^2 + 2hlm) \\
& + y_1y_2(abc_0 + 5ab_0c_1 - ba_2g + 3ab_2n - 9afm + 2a_1b_0c_0 + 7a_1b_2g \\
& \quad - 5a_1c_1h - 4a_2b_0n - 10a_2b_2l + 11a_2fh - 3c_0h^2 - 2a_1fl - 3b_0gl \\
& \quad - 2a_1mn + 4a_2m^2 - 3ghm + 3hln + 6l^2m) \\
& + y_2^2(2acb_0 + ab_2c_0 - 2ca_1h - 3ac_1m - 2a_2b_0c_0 - a_2b_2g + 5a_2c_1h \\
& \quad + 4a_1c_0m - 2a_1c_1l + 4a_1fg - 4a_2fl - 3c_0hl - 4a_1n^2 + 2a_2mn \\
& \quad - 3glm + 6l^2n)] \\
+ \overset{3}{\Sigma} x_0^2x_1^2 [& \overset{2}{\Sigma} y_0^2 \cdot 3(abg - ba_2^2 + 3ab_0n - 2ab_2l + afh - 3am^2 + 2a_1a_2b_2 - a_1^2 f \\
& \quad + a_1b_0g - 4a_2b_0l - 3a_1hn + 4a_2hm + 2a_1lm - 2gh^2 + 2hl^2) \\
& + y_0y_1 \cdot 6(abn + ab_0f + ba_1g - 2ab_2m - 2ba_2l + 2a_2b_2h + 4a_1b_0n \\
& \quad - a_1fh - b_0gh - 4a_1m^2 - 4b_0l^2 - 5h^2n + 10hlm) \\
& + \overset{2}{\Sigma} y_0y_2 \cdot 3(abc_0 + ab_0c_1 - ba_2g - ab_2n - afm + 2a_1b_0c_0 + 3a_1b_2g \\
& \quad - a_1c_1h + 4a_2b_0n - 2a_2b_2l + 3a_2fh - 3c_0h^2 - 2a_1fl - 7b_0gl \\
& \quad - 2a_1mn - 4a_2m^2 + 5ghm - hln + 6l^2m) \\
& + y_2^2(abc - ab_2c_1 - ba_2c_0 + 2ca_1b_0 - 3ch^2 + 4a_1b_2c_0 + 4a_2b_0c_1 - 6a_1c_1m \\
& \quad - 6b_0c_0l - 3a_2fm - 3b_2gl + 3c_0hm + 3c_1hl + 6fgh - 3fl^2 - 3gm^2 \\
& \quad - 6hn^2 + 12lmn)]
\end{aligned}$$

$$+\overset{3}{\Sigma}x_0^2x_1x_2[y_0^2\cdot 3(3ab_0c_0+2ab_2g+2ac_1h-4afl-3amn-2a_1^2c_1-2a_2^2b_2$$
$$+4a_1a_2f-3a_1c_0h-3a_2b_0g+5a_1gm+5a_2hn-2a_1ln-2a_2lm$$
$$-4ghl+4l^3)$$
$$+\overset{2}{\Sigma}y_0y_1\cdot 3(abc_0+3ab_0c_1-ba_2g+ab_2n-5afm+6a_1b_0c_0+5a_1b_2g$$
$$-3a_1c_1h+4a_2b_0n-6a_2b_2l+7a_2fh-7c_0h^2-2a_1fl-13b_0gl$$
$$-6a_1mn-4a_2m^2+9ghm+hln+10l^2m)$$
$$+\overset{2}{\Sigma}y_1^2\cdot 3(abc_1+2ba_1c_0-ab_2f-3bgl+4a_2b_0f-2b_0c_0h-4a_2b_2m$$
$$+5b_2gh-c_1h^2-2a_1fm-2b_0gm-fhl+4lm^2)$$
$$+y_1y_2(abc+8ab_2c_1+2ba_2c_0+2ca_1b_0-9af^2-3bg^2-3ch^2+16a_1b_2c_0$$
$$+16a_2b_0c_1-12a_1c_1m-12a_2b_2n-6a_1fn-6a_2fm-18b_0gn$$
$$-12b_2gl-18c_0hm-12c_1hl+30fgh+6gm^2+6hn^2+24lmn)].$$

This connex may be considered either as a conic or a quartic, depending on whether x or y is the given point. If the curve has a double point the conic vanishes, but the quartic does not. Let

$$a=a_1=a_2=0,$$

so that (1, 0, 0) is a double point, and let this be taken as the point y. Then

$$A^3x^4y^2=\overset{2}{\Sigma}x_0^2x_1^2\cdot 3(-2gh^2+2hl^2)+x_0^2x_1x_2\cdot 3(-4ghl+4l^3)$$
$$+\overset{2}{\Sigma}x_0x_1^3\cdot 3(-b_0gh-h^2n+2hlm)$$
$$+\overset{2}{\Sigma}x_0x_1^2x_2\cdot 3(-c_0h^2-2b_0gl-ghm+4l^2m)$$
$$+\overset{2}{\Sigma}x_1^4(bgh-bl^2-b_0^2g-b_0hn+2b_0lm)$$
$$+\overset{2}{\Sigma}x_1^3x_2(-b_0c_0h+4b_2gh-4b_0gm+2b_0ln-4b_2l^2-3hmn+6lm^2)$$
$$+x_1^2x_2^2(-3b_0gn-3c_0hm+6fgh-6fl^2-3gm^2-3hn^2+12lmn).$$

Obviously this curve has a double point at (1, 0, 0) with the same tangents as the quartic. Therefore six of the sixteen intersections of the two curves are used up here. To find the others let us take (0, 1, 0) on $(\alpha x)^4$, so that $b=0$. Then in order that (0, 1, 0) may also be on $A^3x^4y^2$ we must have

$$b_0(b_0g+hn-2lm)=0.$$

If $b_0=0$, the intersection is the contact of a tangent from the double point; there are six such. If $b_0\neq 0$, let us take another intersection which is not one of these contacts as (0, 0, 1). Then

$$c=0,\quad b_0g+hn-2lm=0,\quad c_0h+gm-2ln=0.$$

These last two equations say that the tangents at the double point, given by $hx_1^2+2lx_1x_2+gx_2^2=0$, are apolar to the three points in which x_0 cuts the polar cubic of the double point, these three points being given by

$$b_0\xi_2^3-3m\xi_1\xi_2^2+3n\xi_1^2\xi_2-c_0\xi_1^3=0.$$

But this is true when x_0 is the flex line of the polar cubic. This result can be verified by starting out merely with $a=a_1=a_2=0$ and throwing the polar cubic of (1, 0, 0),

$$3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3,$$

into the canonical form by

$$h=g=m=n=0.$$

Then x_0 is the flex line. It meets $A^3x^4y^2$ in points given by

$$-l^2(bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)=0,$$

evidently the same points as those in which x_0 meets the quartic. Therefore the four remaining intersections of $A^3x^4y^2$ and $(\alpha x)^4$ are on the flex line of the polar cubic of the double point.

This result can be verified algebraically. For

$$\begin{aligned}A^3x^4y^2=&(gh-l^2)(6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2\\&+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4)\\&-[3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3]\\&\quad[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2]\\=&(gh-l^2)(\alpha x)^4-(\alpha x)^3(\alpha y)\cdot\text{flex line}.\end{aligned}$$

The polar cubic picks up the six intersections at the double point and those at the six contacts of tangents from the double point; the flex line gives the remainder.

The flex line can be obtained by a sort of limit process from the polar lines of $(\alpha x)^4$ and $(hx)^6$. Instead of at once taking (1, 0, 0) as a double point, let us make x_0 the polar line of (1, 0, 0) so that $a_1=a_2=0$. Then if y is (1, 0, 0),

$$\begin{aligned}&3(\alpha y)^4\cdot(hx)(hy)^5+(hy)^6\cdot(\alpha x)(\alpha y)^3\\&\qquad=a[4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2].\end{aligned}$$

If we divide this by $(\alpha y)^4=a$ and then let $a=0$ so as to get a double point, we have the flex line.

$A^3x^4y^2$ is not the only curve which picks up the six points of contact from y when y is a double point. So does $(hx)^5(hy)$, for when the double point is taken as $(1, 0, 0)$,

$$\begin{aligned}(hx)^5(hy) = {} & \overset{2}{\Sigma} x_0^3x_1^2 \cdot 2(-gh^2+hl^2) + x_0^3x_1x_2 \cdot 4(-ghl+l^3) \\ & + \overset{2}{\Sigma} x_0^2x_1^3 \cdot (-b_0gh-2b_0l^2-3h^2n+6hlm) \\ & + \overset{2}{\Sigma} x_0^2x_1^2x_2(-3c_0h^2-6b_0gl+3ghm+6l^2m) \\ & + \overset{2}{\Sigma} x_0x_1^4 \cdot \tfrac{1}{3}(bgh-4bl^2-b_0^2g-4b_0hn+6b_2hl+2b_0lm-3fh^2+3hm^2) \\ & + \overset{2}{\Sigma} x_0x_1^3x_2 \cdot \tfrac{2}{3}(-3bgl-2b_0c_0h+5b_2gh-3c_1h^2-2b_0gm-2b_0ln \\ & \quad -2b_2l^2+3fhl+6lm^2) \\ & + x_0x_1^2x_2^2 \cdot \tfrac{1}{3}(-3bg^2-3ch^2-6b_0c_0l-6b_0gn-6b_2gl-6c_0hm \\ & \quad -6c_1hl+18fgh+18lmn) \\ & + \overset{2}{\Sigma} x_1^5 \cdot \tfrac{1}{3}(bhn-2blm-b_0^2n+2b_0b_2l-b_0fh+b_0m^2) \\ & + \overset{2}{\Sigma} x_1^4x_2 \cdot \tfrac{1}{3}(bc_0h-2bgm-2bln-b_0^2c_0+2b_0b_2g-2b_0c_1h+4b_0fl \\ & \quad +4b_2hn-2b_0mn-2b_2lm-3fhm+3m^3) \\ & + \overset{2}{\Sigma} x_1^3x_2^2 \cdot \tfrac{1}{3}(-cb_0h-3bgn+4b_2c_0h-4b_0c_0m+2b_0c_1l+5b_0fg \\ & \quad -2b_2gm-6c_1hm-2b_0n^2-2b_2ln+3fhn+6m^2n),\end{aligned}$$

and the coefficient of x_1^5 vanishes for $b=b_0=0$. That it passes through the intersections of $(\alpha x)^4$ and $(\alpha x)^3(\alpha y)$ is shown by

$$\begin{aligned}3(hx)^5(hy) = {} & [6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2+12nx_0x_1x_2^2 \\ & +4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4] \\ & \quad [4(gh-l^2)x_0+(b_0g+hn-2lm)x_1+(c_0h+gm-2ln)x_2] \\ & + [3hx_0x_1^2+6lx_0x_1x_2+3gx_0x_2^2+b_0x_1^3+3mx_1^2x_2+3nx_1x_2^2+c_0x_2^3] \\ & \quad [x_0^2 \cdot -10(gh-l^2)+\overset{2}{\Sigma} x_0x_1(-5b_0g-5hn+10lm) \\ & \qquad +\overset{2}{\Sigma} x_1^2(-bg-b_0n+2b_2l-fh+m^2) \\ & \qquad\quad +x_1x_2(-b_0c_0-2b_2g-2c_1h+4fl+mn)] \\ = {} & (\alpha x)^4 \cdot \text{flex line} - (\alpha x)^3(\alpha y)\left[\frac{1}{a}A^3x^2y^4 + x_0 \cdot \text{flex line}\right],\end{aligned}$$

where by $\frac{1}{a}A^3x^2y^4$ is meant the result of forming $A^3x^2y^4$ for $y\equiv(1, 0, 0)$ when only $a_1=a_2=0$, dividing by a, and then letting $a=0$. Any further pursuit of this relation, however, when the quartic has no double point, seems merely to lead to Cayley's indentity

$$3(\alpha x)^4 \cdot (hx)(hy)^5 - (\alpha x)^3(\alpha y) \cdot A^3x^2y^4 + (\alpha x)(\alpha y)^3 \cdot A^3x^4y^2 \\ -3(\alpha y)^4 \cdot (hx)^5(hy) \equiv 0.$$

Note 1.

The developments of H and S are given by Salmon only for a special quartic. Those for the general quartic may be calculated by means of a differential operator from the leading coefficients, which in case of S is obtained directly from the invariant S of the cubic as given in the *Higher Plane Curves.* The general expression for the Hessian is

$$
\begin{aligned}
&\overset{3}{\Sigma} x_0^6(agh-al^2-a_1^2g-a_2^2h+2a_1a_2l)\\
&+\overset{6}{\Sigma} x_0^5x_1\cdot 2(ab_0g+ahn-2alm-a_2^2b_0-a_1^2n+2a_1a_2m-a_1gh+a_1l^2)\\
&+\overset{6}{\Sigma} x_0^4x_1^2(abg-ba_2^2+4ab_0n-2ab_2l+afh-4am^2+2a_1a_2b_2-a_1^2f+2a_1b_0g-6a_2b_0l\\
&\qquad -4a_1hn+6a_2hm+2a_1lm-3gh^2+3hl^2)\\
&+\overset{3}{\Sigma} x_0^4x_1x_2\cdot 2(2ab_0c_0+ab_2g+ac_1h-2afl-2amn-a_1^2c_1-a_2^2b_2+2a_1a_2f-2a_1c_0h\\
&\qquad -2a_2b_0g+4a_1gm+4a_2hn-2a_1ln-2a_2lm-3ghl+3l^3)\\
&+\overset{3}{\Sigma} x_0^3x_1^3\cdot 2(abn+ab_0f+ba_1g-2ab_2m-2ba_2l+2a_2b_2h+2a_1b_0n-a_1fh-b_0gh\\
&\qquad -2a_1m^2-2b_0l^2-3h^2n+6hlm)\\
&+\overset{6}{\Sigma} x_0^3x_1^2x_2\cdot 2(abc_0+2ab_0c_1-ba_2g-3afm+2a_1b_0c_0+4a_1b_2g-2a_1c_1h+2a_2b_0n\\
&\qquad -4a_2b_2l+5a_2fh-3c_0h^2-2a_1fl-6b_0gl-2a_1mn-2a_2m^2+3ghm+6l^2m)\\
&+x_0^2x_1^2x_2^2(abc+2ab_2c_1+2ba_2c_0+2ca_1b_0-3af^2-3bg^2-3ch^2+10a_1b_2c_0+10a_2b_0c_1\\
&\qquad -6a_1c_1m-6a_2b_2n-6b_0c_0l-6a_1fn-6a_2fm-6b_0gn-6b_2gl-6c_0hm-6c_1hl\\
&\qquad +18fgh+18lmn).
\end{aligned}
$$

The full development for S is

$$
\begin{aligned}
&\overset{3}{\Sigma} x_0^4(ab_0c_0l-ab_0gn-ac_0hm+agm^2+ahn^2-almn-a_1a_2b_0c_0+a_1^2c_0m+a_2^2b_0n\\
&\qquad +a_1b_0g^2+a_2c_0h^2-a_1c_0hl-a_2b_0gl-a_1^2n^2+a_1a_2mn-a_2^2m^2+a_1ghn+a_2ghm\\
&\qquad -3a_1glm-3a_2hln+2a_1l^2n+2a_2l^2m-g^2h^2+2ghl^2-l^4)\\
&+\overset{6}{\Sigma} x_0^3x_1(abc_0l-abgn-ba_1a_2c_0-ab_2c_0h+ab_0c_1l+ba_2^2n-ab_0fg+ba_1g^2+2ab_2gm\\
&\qquad -ac_1hm-ba_2gl-ab_2ln+2afhn-aflm+a_1^2b_2c_0-a_1a_2b_0c_1+a_2^2b_0f+a_2b_0c_0h\\
&\qquad +a_1^2c_1m+a_1a_2b_2n-2a_2^2b_2m-a_1b_0c_0l+a_2b_2gh+a_2c_1h^2-2a_1^2fn+a_1a_2fm\\
&\qquad +2a_1b_0gn-3a_1b_2gl-a_1c_1hl+2a_2b_2l^2-2a_2b_0ln+a_1fgh-b_0g^2h-3a_2fhl\\
&\qquad +2a_1fl^2-2a_1gm^2-a_2hmn+b_0gl^2+a_1lmn+2a_2lm^2-gh^2n+2ghlm\\
&\qquad +hl^2n-2l^3m)\\
&+\overset{3}{\Sigma} x_0^2x_1^2(abc_0m+abc_1l-abfg-abn^2-ab_0b_2c_0-ba_1a_2c_1+ab_2^2g-ab_2c_1h+ba_2^2f\\
&\qquad -ba_2c_0h+ab_0fn-ab_2fl+ba_1gn-ba_2gm+ab_2mn+ba_2ln+af^2h+bg^2h\\
&\qquad -afm^2-bgl^2+a_1^2b_2c_1-a_2^2b_2^2+a_2b_0^2c_0+a_1a_2b_2f+a_1b_2c_0h+a_2b_0b_2g+a_2b_0c_1h\\
&\qquad -a_1b_0c_0m-a_1b_0c_1l-a_1^2f^2-b_0^2g^2-a_1b_2gm-a_2b_0fl+2a_2b_2hn+2a_1b_0n^2-3a_1b_2ln\\
&\qquad -3a_2b_0mn-a_1fhn-2a_2fhm-b_0ghn-2b_2ghl+4a_1flm+4b_0glm-a_1m^2n\\
&\qquad +2a_2m^3-b_0l^2n+2b_2l^3+fgh^2-fhl^2-ghm^2-h^2n^2+4hlmn-3l^2m^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\overset{3}{\Sigma} x_0^2 x_1 x_2 (abcl - bca_1a_2 - abc_1g - acb_2h + ba_2^2c_1 + ca_1^2b_2 - 2ab_0c_0f + ba_1c_0g \\
&\quad + ca_2b_0h + 2ab_0c_1n + 2ab_2c_0m - ba_2c_0l - ca_1b_0l + ab_2fg + ac_1fh - 2ab_2n^2 \\
&\quad - 2ac_1m^2 - af^2l + 2afmn - a_1^2c_1f - a_2^2b_2f - a_1b_0c_1g - a_2b_2c_0h + 2a_1b_0c_0n - a_1b_2c_0l \\
&\quad + 2a_2b_0c_0m - a_2b_0c_1l + a_1a_2f^2 + 2a_1c_0fh + 2a_2b_0fg - b_0c_0gh - a_1b_2gn - 2a_1c_1hn \\
&\quad - 2a_2b_2gm - a_2c_1hm - 4a_1c_0m^2 + 5a_1c_1lm - 4a_2b_0n^2 + 5a_2b_2ln - b_0c_0l^2 + 2b_2g^2h \\
&\quad + 2c_1gh^2 + a_1fgm + a_2fhn - 2b_0g^2m - 2c_0h^2n - 3a_1fln - 3a_2flm + 4b_0gln \\
&\quad - 2b_2gl^2 + 4c_0hlm - 2c_1hl^2 + 2a_1mn^2 + 2a_2m^2n - 4fghl - 3ghmn + 4fl^3 \\
&\quad + 4glm^2 + 4hln^2 - 7l^2mn).
\end{aligned}
$$

The leading coefficient of T may be obtained from Salmon's expression for the invariant T of the cubic, but the coefficients are too tedious to compute in general. For the coefficients of s and t see AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIX (1917), p. 232.

Note 2.

Certain invariants of the quartic have a well-defined geometrical meaning. The two simplest invariants, the A^3 and A^6, have apolarity meanings and may be included in the list by courtesy. Then comes the A^{15} of Professor Coble,* which is the condition that the quartic be reducible to the sum of the fourth powers of the six lines of a complete quadrilateral, also that the covariant S become two conics; the discriminant, A^{27}; the A^{48} of Dr. Thomsen, expressing that there be a polar conic which is the square of a line; an A^{54} of Professor Morley's expressing the condition that the quartic pass through the vertices of a pentagon; the undulation condition, A^{60}. In this list certain gaps are filled in by the A^{45}, the condition that a polar cubic be made up of three lines, and the A^{51}, the condition that the polar conic of two points be the square of their join.

There should also be an A^{24} under which the Steinerian has the stationary lines of the quartic as double lines. For the Steinerian in lines, an $A^{36}\xi^{18}$, and the Cayleyan, an $A^{12}\xi^{18}$, touch on the stationary lines.† Then the terms of the Steinerian containing only the first power of $(s\xi)^4$ or $(t\xi)^6$ should be the same as those of the Cayleyan, multiplied by an A^{24} to bring them up to the proper degree. If this A^{24} vanishes, then Σ has the stationary lines as double lines.

---

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 357.

† Proc. Nat. Ac. Sci., Vol. III (1917), p. 449.